Estimation of Distributed Parameter Systems: Some Closed-Form Solutions

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The estimation and control of distributed parameter systems has numerous applications in the control of structures. In many cases, obtaining the solutions of estimation and control problems that involve partial differential equations requires the use of numerical methods and yields only approximate answers. In this paper, exact solutions are obtained for several practical estimation problems associated with static systems, solutions that cannot be obtained by any other reasonable approach.

Introduction

In recent years, there has been increasing interest in large space structures that require precise figure control, attitude control, and pointing control.[1] The advent of the NASA Space Shuttle has opened the way to launching deployable or erectable satellites that are an order of magnitude larger and more complex than previous satellites. This capability makes possible new applications in Earth sensing, communications, astrophysics, and detection and ranging of flight vehicles. These spacecraft will require new techniques for control. It will no longer be sufficient to design a stiff structure and control it as a rigid body. Large antennas and mirrors require precise figure control and augmented damping and stiffness of the structure.

In an earlier work, Schaechter[2] investigated the estimation of dynamic systems governed by partial differential equations. The results for the estimator gains took the form of square root integral operators for which no closed-form expressions could be found. Rodriguez[3] has also investigated this problem and has obtained numerical results for the estimation of the attitude of a flexible beam. Weeks[4] has developed a Green's functions approach to the estimation of static systems. It is this approach that will be followed in this paper for obtaining closed-form solutions for several systems.

The Motivating Problem

The real-world problem that motivated this study is presented below. Given a set of noisy measurements of the displacement (or slope) of a two-dimensional surface, find the least-squares estimate of the shape of the surface consistent with the fact that the surface is smooth. This is simply a weighted least-squares problem whose solution is readily available.[5] With a notation that will be used throughout the remainder of this paper, the problem can be formulated mathematically as follows.

Given:

$$z = Hu + v \tag{1}$$

measurements of the surface displacement, u, corrupted by measurement noise, v

$$Lu = w$$

a system model reflecting smoothness, i.e., the relation between the process disturbance force, w, and the displacement, u.

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Find: u that minimizes the cost functional J, where

$$J = (z - Hu)'R^{-1}(z - Hu) + u'L'Q^{-1}Lu \tag{2}$$

where R and Q are the covariances of the measurement and process noises respectively. The simple solution to this problem is

$$u = (H'R^{-1}H + L'Q^{-1}L)^{-1}H'R^{-1}z \tag{3}$$

As stated earlier, the measurement z consists of the displacements (slopes) of a two-dimensional surface. A relatively modest number of such measurements is available for the estimation process; in this problem the actual value is that z is a 32×32 matrix of measured displacements at points corresponding to a 32×32 grid on the two-dimensional surface. From this, the object is to obtain an improved real-time estimate of the displacement of the surface at each of the measurement locations. In fact, this problem has a 32×32 ($= 1024$) dimensional state vector u and an identically sized measurement vector z. The covariance matrix that needs to be computed (though only once) for the solution of the least-squares problem is therefore a 1024×1024 matrix! The fact that such an apparently simple problem can lead to such astonishing difficulty is what motivated a look at this problem from a continuum viewpoint.

Continuum Modeling

Let us now turn our attention to the advantages of reformulating the previous problem in terms of differential operators. Although this step may seem to be quite formidable, there is very little additional work that needs to be done—just some new interpretations of the previous results.

In Eq. (1), take L and H to be arbitrary linear operators. This is a general enough statement to allow L and H to be matrices as well as differential operators. For the matrix formulation, these two notions are the same. For the case of the differential operator, the variable u is a scalar function, and in this example, it is a function of the two spatial variables defining the domain of interest. *The fact that a single scalar variable can replace an arbitrarily large state vector is the key to the simplifications that can be obtained with differential operators.* The performance measure J becomes defined in terms of an integral (rather than a large summation) of the scalar errors over the appropriate domain. Along with these changes in Eqs. (1) and (2), we replace the matrix transpose operation in Eqs. (2) and (3) with the adjoint operation. The final association that must be made in order to complete this transition is that of the matrix inverse in Eq. (3). The inverse of a differential operator is an integral operator whose kernel is the Green's function G, associated with the differential operator and its boundary conditions. With these changes the Green's function $G(x,\xi)$ now has the interpretation of the covariance of the state estimate, and the entire expression that multiplies z in Eq. (3) can be interpreted as the optimal estimator gain. In Eq. (3), the expression for the estimated state becomes

$$u(x) = \int_\Omega G(x,\xi)z(\xi)\mathrm{d}\Omega \tag{4}$$

In the remainder of this paper, examples for which *closed-form* solutions can be found will be presented, including the problem posed earlier.

Examples

In this section, four estimation examples will be presented. The common features in these examples are that they are all modeled as continua, and they all have exact, closed-form solutions. The examples are as follows:

1) Estimation of the spatially continuous displacement of a one-dimensional string with fixed ends from a spatially continuous but noisy displacement measurement.

2) Estimation of the spatially continuous displacement of a two-dimensional membrane with fixed ends from a spatially continuous but noisy displacement measurement.

3) Estimation of the spatially continuous displacement of a one-dimensional string with a spatially continuous but noisy *slope* measurement.

4) Estimation of the spatially continuous shape of a two-dimensional membrane with a spatially continuous but noisy *slope* measurement.

Example 1 will be treated in somewhat more detail than the remainder of the examples so that the procedure followed can be made clear. Figure 1 shows a picture of the system that will now be considered. This system consists of a tensioned flexible string of unit length, with the displacements at the ends constrained to be zero.

The differential equation and boundary conditions for this system that relate the displacements of the string to the transverse force inputs are

$$u'' = w, \qquad u(0) = u(1) = 0 \tag{5}$$

where $(\ )''$ denotes the second derivative with respect to the spatial variable. The noisy displacement measurement is modeled as

$$z = u + v \tag{6}$$

Comparing Eqs. (5) and (6) with Eqs. (1-3) gives

$$H = 1 \tag{7}$$

$$L = \langle\ \ \rangle'' \tag{8}$$

$$J = \int_0^1 [(z-u)^2/R + (\partial u/\partial x)^2/Q]\mathrm{d}x \tag{9}$$

$$u = (R/Q\ (\ )'''')^{-1}z \tag{10}$$

All that remains to be found is the Green's function for the differential operator given in Eq. (10). Clearly this Green's function $G(x,\xi)$ must satisfy the following differential equation:

$$\frac{R}{Q}G''''(x,\xi) + G(x,\xi) = \delta(x-\xi) \tag{11}$$

where $\delta(x-\xi)$ is the Dirac delta and where the boundary conditions on $G(x,\xi)$ must satisfy

$$G(0,\xi) = G(1,\xi) = G(x,0) = G(x,1) = 0$$

$$G''(0,\xi) = G''(1,\xi) = G''(x,0) = G''(x,1) = 0 \tag{12}$$

These boundary conditions arise from the boundary conditions on u, plus the additional boundary conditions that result from the integration by parts of the cost functional in Eq. (9). After some effort, the exact and closed-form solution for G is obtained. It is worthwhile, however, before presenting this solution, to look at the results for the limiting values of the quantity Q/R. For the case $R=0$, i.e., a perfect measurement of the displacement of the string, the solution of Eqs. (11) and (12) is just that $G(x,\xi) = \delta(x-\xi)$. This result is sketched in Fig. 2a for a particular value of ξ. Then

$$u(x) = \int_0^l G(x,\xi)z(\xi)\mathrm{d}\xi = \int_0^l \delta(x-\xi)z(\xi)\mathrm{d}\xi = z(x) \tag{13}$$

The result in Eq. (13) is just what one would expect. In the case of a *perfect* measurement of the displacement of the string, the best estimate of that displacement is just the measurement itself. At the other extreme, for large R/Q, the fourth derivative of $G(x,\xi)$ is discontinuous; i.e., $G(x,\xi)$ is very smooth and satisfies the boundary conditions in Eq. (12). As such, it must have a shape roughly like the one shown in Fig. 2b.

Evidently, the estimator gain, i.e., the true Green's function, is just a function of R/Q. The Green's function must behave like a delta function for R/Q small, and for large R/Q, the weighting must spread to make use of measurements adjacent to ξ_0 to improve the estimate of the displacement at ξ_0. The exact expression for $G(x,\xi)$ is given in Eq. (14). At first glance, Eq. (14) is apparently of little practical value. However, it is much easier to *evaluate* Eq. (14) at N discrete points than to *invert* an $N \times N$ matrix, particularly when N is large. Figure 3 shows plots of this Green's function at 32 discrete spatial locations as R/Q varies. Notice in Fig. 3a how the Green's function is approximately a delta function. Professor A. E. Bryson of Stanford University has pointed out that these results may be interpreted as a noncausal Kalman

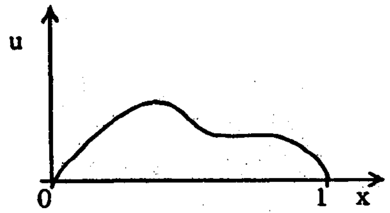


Fig. 1 The one-dimensional string.

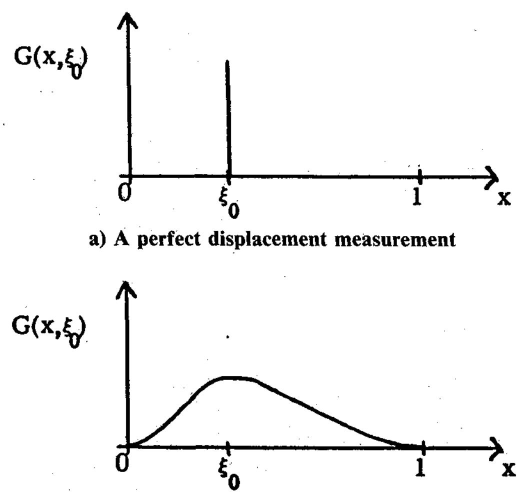


Fig. 2 Estimator gains for the continuous string.

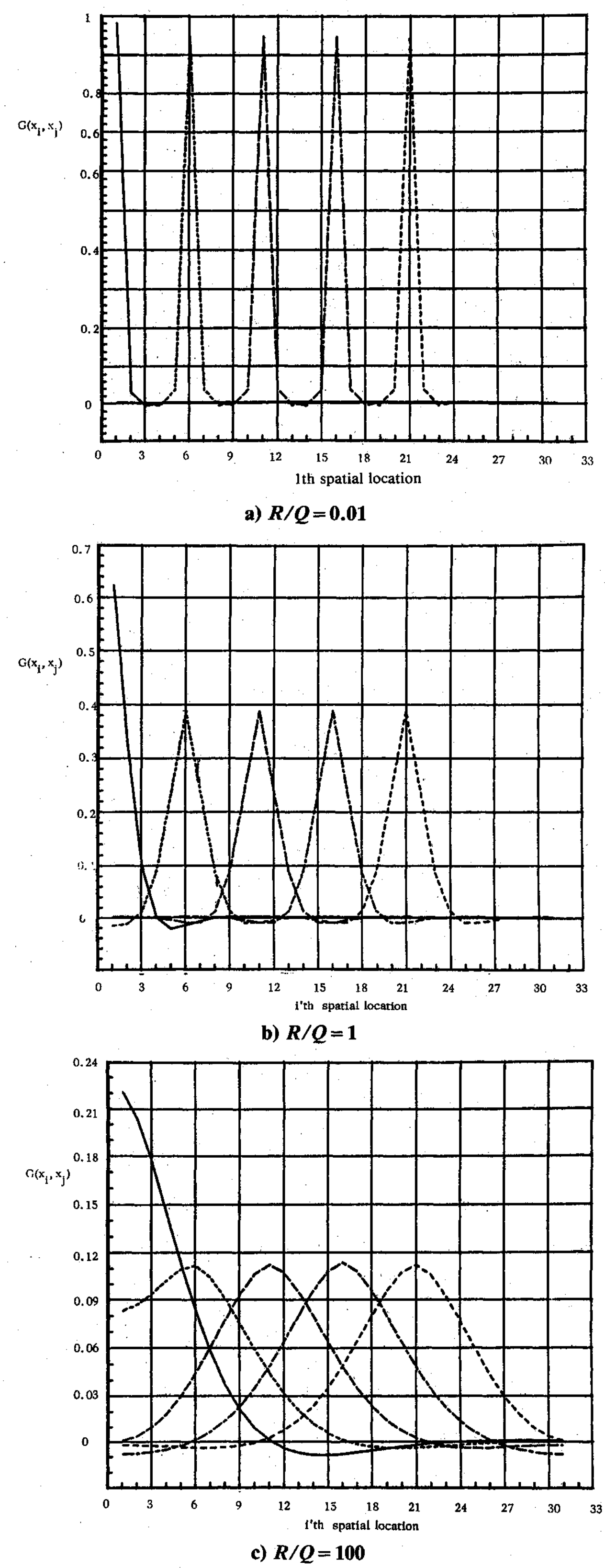


Fig. 3 The exact Green's function for the continuous string.

filter, in that measurements before and after the spatial point x_i are used to improve the estimate at x_i. Depending on the actual noises present, one particular Green's function $G(x_i, x_j)$ [or $G(x,\xi)$] can be used for the estimation problem of interest. It is also noteworthy that for the case of good measurements, such as the one shown in Fig. 3a, the Green's function is roughly the same from spatial point to spatial point, and it takes on nonzero values only at nearby points. This feature

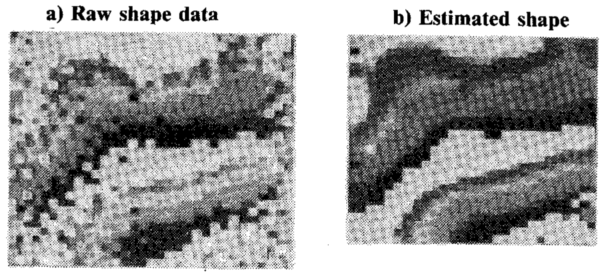


Fig. 4 Estimation of the shape of a two-dimensional surface.

has been exploited in digital computer applications of this algorithm to reduce the computational load of its implementation. Another way of thinking of this is that far from the initial or final states, the noncausal Kalman filter reaches a statistical steady state.

$$G(x,z) = \left[-\frac{\cos(qz+qx)\sinh(qz+qx-2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} - \frac{\cosh(qz+qx)\sin(qz+qx-2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} + \frac{\sinh(qz+qx)\cos(qz+qx-2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} - \frac{\cosh(qz-qx)\sin(qz-qx+2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} + \frac{\sinh(qz-qx)\cos(qz-qx+2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} + \frac{\sin(qz+qx)\cosh(qz+qx-2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} - \frac{\cos(qz-qx)\sinh(qz-qx+2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} + \frac{\sin(qz-qx)\cosh(qz-qx+2q)}{8q^3\cosh(2q)-8q^3\cos(2q)} \right] \cdot \frac{q}{2} \tag{14}$$

and

$$q^4 = Q/R$$

The second example to be discussed is the two-dimensional analog of the previous example. In this case, $L = R/Q\nabla^2$, where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{15}$$

If the spatial region of interest is the unit square $0<x<1$, $0<y<1$, and the boundary conditions are such that the deflection of the membrane is constrained to be zero along the boundary of the region, the solution to the estimation problem is obtained by finding the Green's function corresponding to the differential equation

$$\frac{R}{Q}\nabla^4 G(x,y,\xi,\eta) + G(x,y,\xi,\eta) = \delta(x-\xi, y-\eta) \tag{16}$$

The analysis of this seemingly more complex problem is hastened by recognizing that the differential equation and the boundary conditions for this problem are completely symmetric in the x and y spatial directions. This leads to a guess for the Green's function that is the product of Green's functions from the first example in the x and y directions. This is the result that was sought for the original problem of estimating the true shape of a smooth two-dimensional surface from a 32×32 array of noisy displacement measurements from across the surface. The results of this procedure, when applied to real data, are shown in Fig. 4. Notice how the filtered surface is much cleaner than the raw data. It should be remembered that the identical problem, if solved with a conventional least-squares procedure, would require the inversion of a 1024×1024 matrix.

The last two examples to be presented here contain features that do not appear in the first two examples. The common thread through these last two examples is that *slope* measurements are used to deduce the *displacements* of one- and two-dimensional surfaces. Of course, in these examples it will be possible to determine the displacements only up to an arbitrary constant of integration. Alternatively, we may specify the displacement of any one point on the surface to remove the ambiguity. The second, and somewhat more subtle, new feature in these examples is that no model of the force-displacement characteristics of the structure is *needed* to obtain results. This is distinctly different from the previous examples and from the usual least-squares problem, and so this topic warrants further discussion.

The use of a system model yields additional information in the optimization process. When the system model consists of a differential operator (in time, or in space), the resulting solution to the least-squares problem is guaranteed to be of a certain smoothness consistent with the model. *When a spatially continuous slope measurement of a system is made, however, a continuous solution is guaranteed without the use of a differential equation model.* The reason for this is that when slope measurements are made, an integration is required in order to obtain the displacements. This integration forces the estimate to be continuous, even if the measurement is noisy. From a mathematical perspective, only one of the terms in the inverse operator appearing in Eq. (3) needs to be a differential operator for the resulting estimate to be continuous. In the next example, the slope measurement ($H = \partial/\partial x$) will serve this purpose. Since the following two examples in one and two spatial dimensions make use of no model information and use only slope measurements, they may be thought of as optimal integrators for their respective dimensions.

The third example is the problem of estimating the spatially continuous shape of a one-dimensional string with a spatially continuous slope measurement. Figure 5 shows a schematic of this system. In Eq. (3), we set $L = 0$ and $H = \partial/\partial x$ to give

$$u(x) = -\left(\frac{\partial^2}{\partial x^2}\right)^{-1}\frac{\partial z}{\partial x} = \int_0^1 G(x,\xi)\frac{\partial z(\xi)}{\partial \xi}\mathrm{d}\xi \tag{17}$$

With the preceding arguments, we will rely on the measurements alone, and may arbitrarily take $u(0) = 0$. Obtaining the closed-form result for this problem is straightforward:

$$\begin{aligned} G(x,\xi) &= -x, \quad x<\xi \\ &= -\xi, \quad x>\xi \end{aligned} \tag{18}$$

In Eq. (17), it certainly does not seem reasonable to differentiate the noisy measurements z in order to obtain estimated

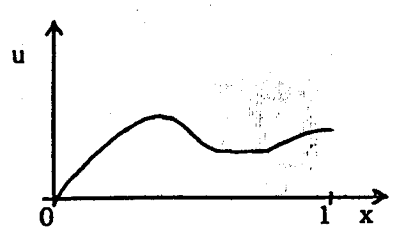


Fig. 5 The one-dimensional string with slope measurement.

displacements from the slope measurements and, in fact, this differentiation is not required. We may integrate Eq. (17) by parts to obtain the more desirable result

$$u(x) = -\int_0^1 \frac{\partial}{\partial \xi}G(x,\xi) \quad z(\xi)\mathrm{d}\xi \tag{19}$$

where

$$\begin{aligned} \frac{\partial}{\partial \xi}G(x,\xi) = \quad & 0, \quad x<\xi \\ & -1, \quad x>\xi \end{aligned}$$

so

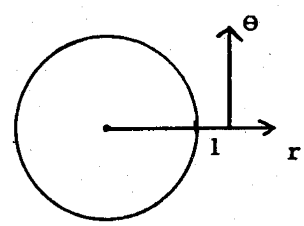
$$u(x) = \int_0^x z(\xi)\mathrm{d}\xi \tag{20}$$

The results in Eqs. (18-20) should come as no surprise. Equation (18) means that if we start with perfect state information ($u(0) = 0$) and have a unit variance measurement of the slope of u, then the estimate of the displacement at x is correlated with the estimate at ξ to the extent that the measurements overlap. It should also be clear that slope measurements at ξ, which are further away from the origin than the point x, can in no way improve the determination of the displacement at x. Also, $G(x,x)$, which is the variance of the estimate at x, grows linearly with x. The result in Eq. (20) shows that, given the slope measurements z, even if they are noisy, the best way to obtain the displacement at x is to integrate the measurements. Again, this is not too surprising a result.

The final example to be presented is a two-dimensional analog of the previous example. In this case, the domain of interest is the unit circular region shown in Fig. 6. The slope measurements consist of both the radial and the circumferential spatial derivatives of the displacements of the surface. We arbitrarily fix $u(0,0) = 0$. It is desired to estimate the displacement of the surface from the slope measurements.

As a preface to the somewhat surprising results obtained in this example, consider a simplified spatially discrete version of

Fig. 6 The unit circular disk.



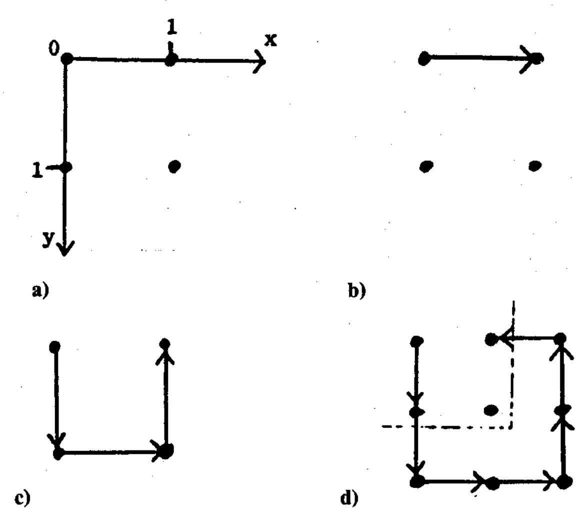


Fig. 7 A simplfied two-dimensional problem. a) 2×2 grid; b) 1st estimate of *u* (1,0); c) 2nd estimate of *u* (1,0); d) improved estimate.

this problem. Refer to Fig. 7. The simplified two-dimensional model consists of the 2×2 grid of points shown in Fig. 7a. Given unit variance measurements of the x and y slopes at these four points and the fact that $u(0,0)=0$, what is $u(1,0)$? And how accurately is it determined? Figure 7b shows how one estimate of $u(1,0)$ is obtained from the x slope measurement at (0,0). The variance of the resulting estimate is 1. Figure 7c shows that there is a second, independent determination of $u(1,0)$. It is a longer path, so the variance of this estimate is 3. These two independent determinations of $u(1,0)$ can be statistically combined to produce the best estimate of $u(1,0)$. The variance of this estimate is 3/4, which is a better result than that obtained by only a one-dimensional integration. Furthermore, additional slope measurements from past the point (1,0) improve the estimate of the displacement at this point as is shown in Fig. 7d. Since the path in Fig. 7d is only partially correlated with the paths from the smaller domain, some additional information on $u(1,0)$ is obtained. It is found that extending the two-dimensional domain produces a limiting value for the variance of $u(1,0)$ of 0.6980.

Let us now return to the final example. In this case,

$$z_1=\frac{\partial u}{\partial r},\ z_2=\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{21}$$

From Eq. (3) we obtain the best estimate of $u(r,\theta)$ in terms of the measurements as

$$r\nabla^2 u(r,\theta)=\frac{\partial r z_1}{\partial r}+\frac{\partial z_2}{\partial \theta} \tag{22}$$

where $u(0,\theta)=0$. As in the previous case, we require the Green's function for the operator in Eq. (22). It is found to be

$$G(r,\rho,\theta,\beta)=\log[r^2+\rho^2-2r\rho\cos(\beta-\theta)]$$
$$+\log[1+(r\rho)^2-2r\cos(\beta-\theta)]-2\log(r\rho) \tag{23}$$

This result is similar in character to results in Ref. 6. With the association that the Green's function represents the covariance matrix, we find that the variance of the estimate of the deflection at a point (r,θ) grows only as $\ln r$, as opposed to with r in the one-dimensional case. As in the one-dimensional case, it is undesirable and unnecessary to perform the differentiation of the measurements in Eq. (22), so an integration by parts is performed to yield

$$u(r,\theta)=\int\left(\frac{\partial G}{\partial \rho}z_1+\frac{1}{\rho}\frac{\partial G}{\partial \beta}z_2\right)\rho\mathrm{d}\rho\mathrm{d}\beta \tag{24}$$

This is the formula for optimal integration in a two-dimensional circular region.

Conclusions

A differential operator approach has been used to obtain closed-form solutions to estimation problems. In some cases, this approach yields results while numerical methods would not have worked due to the high order of the discretized variable. An example of a two-dimensional estimation problem has been presented and has shown that this approach is easily implemented and produces excellent results.

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References

[1]Balas, M. J., "Trends in Large Space Structures Control Theory: Fondest Hopes, Wildest Dreams," *IEEE Transactions on Automatic Control,* Vol AC-27, June 1982, pp. 535-552.

[2]Schaechter, D. B., "Estimation of Distributed Parameter Systems," *Journal of Guidance, Control, and Dynamics,* Jan.-Feb. 1982, p. 22.

[3]Rodriguez, G., "Model Error Estimation for Uncertain Dynamical Systems," Joint Automatic Control Conference, San Francisco, CA. Aug. 1980.

[4]Weeks, C. J., "The Control and Estimation of Large Space Structures," Joint Automatic Control Conference, Aug. 1980, San Francisco, CA.

[5]Bryson, A. E. and Ho, Y. C., *Applied Optimal Control*, Blaisdell, Waltham, MA 1969.

[6]Myint-U, T., *Partial Differential Equations of Mathematical Physics,* North Holland, New York, pp. 287-290.